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RUNGE-KUTTA METHOD FOR FUZZY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT
In this work, we use Runge-Kutta method of order four for solving of fuzzy Volterra integro-differential equations (FVIDE). We give some numerical examples to illustrate the theory.

KEYWORDS: Fuzzy integrals, fuzzy Volterra integro-differential equations, Runge-Kutta method of order four.

INTRODUCTION
The concept of fuzzy set was first introduced by Zadeh [30]. Since then, the theory has been developed and it is now emerged as an independent branch of Applied Mathematics. The elementary fuzzy calculus based on the extension principle and the integration of fuzzy function was first studied by Dubois and Prade [6]. Goetschal and Voxman [9] preffered a Rieman integral type approach and Kaleva [21] defined the integral of fuzzy functions using the Lebesgue type concept for integration. Lakshmikantham and Mohapatra [12] studied fuzzy integral equations with arbitrary kernals. The properties of fuzzy integral equations was discussed by many researchers[4, 7, 20, 22, 24, 26, 28]. The existence and uniqueness theorems for certain FVIDE involving fuzzy set valued mappings was studied by Hajighasemi et. al. [17]. Numerical procedures for solving fuzzy integral equations with arbitrary kernals have been investigated by Friedman et.al.[15]. The theory of general Runge-Kutta methods for Volterra integro-differential equations of second kind was discussed by Lubich [13]. Mashaallah et. al. [16] used variational iteration method for finding the numerical solution of FVIDE. In this work, we use Runge-Kutta method of order four for solving FVIDE of the second kind.

The structure of this paper is organized as follows. In section 2, we collect some basic concepts and preliminary results. In section 3, we study the FVIDE. In section 4, we discuss Runge-Kutta method of order four for finding the numerical solution of FVIDE. Finally in section 5, the proposed method is illustrated by solving some numerical examples.

PRELIMINARIES
In this section, we give the fundamental notations of fuzzy set theory which is used throughout this paper.

Definition 2.1.

A fuzzy number is a fuzzy set $u : R^1 \to E^1 = [0,1]$ which satisfies

i. u is upper semi-continuous;
ii. $u(x) = 0$ outside some interval $[a,b]$;
iii. There are real numbers b and c, $a \le b \le c \le d$, for which
    1) $u(x)$ is monotonically increasing on $[a,b]$,
    2) $u(x)$ is monotonically decreasing on $[c,d]$,
    3) $u(x) = 1, \quad b \le x \le c.$

The set of all the fuzzy numbers is denoted by E^1 is given by Kaleva [21].

Definition 2.2.

A fuzzy number u is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$, $0\le\alpha\le 1$, which satisfy the following conditions:

i) $\underline{u}(\alpha)$ is a bounded monotonically increasing, left continuous function on $(0,1]$ and right continuous at 0;

ii) $\overline{u}(\alpha)$ is a bounded monotonically decreasing, left continuous function on $(0,1]$ and right continuous at 0;

iii) $\underline{u}(\alpha)\le\overline{u}(\alpha),\quad 0\le\alpha\le 1,$

A crisp number α is simply represented by $\underline{u}(\alpha)=\overline{u}(\alpha)=\alpha,\quad 0\le\alpha\le 1.$ This fuzzy number space as shown in [29], can be embedded into the Banach space $B=\overline{C}[0,1]\times\overline{C}[0,1].$

For arbitrary $u=(\underline{u}(\alpha),\overline{u}(\alpha)),\ \ v=(\underline{v}(\alpha),\overline{v}(\alpha))$ and $k\in R$ we define addition and multiplication by k as

$$(\underline{u+v})(\alpha)=(\underline{u}(\alpha)+\underline{v}(\alpha)),\quad (\overline{u+v})(\alpha)=(\overline{u}(\alpha)+\overline{v}(\alpha)),\qquad if\qquad k\ge 0,$$

$$\underline{ku}(\alpha)=k\underline{u}(\alpha),\quad \overline{ku}(\alpha)=k\overline{u}(\alpha),\qquad if\qquad k\ge 0.$$

Definition 2.3.

A function $F:(a,b)\to E^1$ is called H-differentiable at $t\in(a,b)$ if, for $h>0$ sufficiently small, there exist the H-differences $F(t+h)-F(t),\quad F(t)-F(t-h)$ and an element $F'(t)\in E^1$ such that :

$$\lim_{h\to 0^+} D\left(\frac{F(t+h)-F(t)}{h},F'(t)\right)=\lim_{h\to 0^+} D\left(\frac{F(t+h)-F(t)}{h},F'(t)\right)=0.$$

Then $F'(t)$ is called the fuzzy derivative of F at t.

Definition 2.4.

Let $u,v\in E^1$. If there exist $w\in E^1$ such that $u=v+w$ then w is called the H-difference of u,v and it is denoted by $u-v$.

Definition 2.5.

For arbitrary fuzzy numbers u,v we use the distance [9]:

$$D(u,v)=\sup_{0\le\alpha\le 1}\max\{\,|\underline{u}(\alpha)-\underline{v}(\alpha)|,|\overline{u}(\alpha)-\overline{v}(\alpha)|\,\}$$

and it is shown that (E^1,D) is a complete metric space [25].

Definition 2.6.

Let $F:[a,b]\to E^1$. For each partition $P=\{t_0,t_1,\ldots,t_n\}$ of $[a,b]$ and for arbitrary $\xi_i\in[t_{i-1},t_i],\ 1\le i\le n$ suppose

$$R_P=\sum_{i=1}^{n}F(\xi_i)(t_i-t_{i-1}),\qquad \Delta:=\{|\,t_i-t_{i-1}\,|,\quad i=1,\ldots,n\}.$$

The definite integral $F(t)$ over $[a,b]$ is $\int_a^b F(t)dt=\lim_{\Delta\to 0}R_P$, provided that this limit exists in the metric D [8,9].

If the fuzzy function $F(t)$ is continuous function in the metric D, its definite integral exists [9]. Also,

$$\overline{\left(\int_a^b F(t;\alpha)dt\right)} = \int_a^b \overline{F}(t;\alpha)dt, \qquad \underline{\left(\int_a^b F(t;\alpha)dt\right)} = \int_a^b \underline{F}(t;\alpha)dt.$$

Definition 2.7.

Let $F:[a,b]\to E^1$ be differentiable. Denote $[F(t)]^\alpha = [f(t;\alpha), g(t;\alpha)]\ \alpha\in[0,1]$. Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'(t;\alpha), g'(t;\alpha)]\ \alpha\in[0,1]$.

Lemma 2.1.

If f and g are Henstock integrable functions and if the function given by $D(f(t), g(t))$ is Lebesgue integrable, then

$$D\left((FH)\int_a^b f(t)dt, (FH)\int_a^b g(t)dt\right) \le (L)\int_a^b D(f(t), g(t))dt.$$

Definition 2.8.

Let $f:[a,b]\to E^1$ be a bounded function. Then the function

$$\varphi_{[a,b]}(f,\delta) = \sup\{D(f(x), f(y));\quad x,y\in[a,b], |x-y|\le\delta\}$$

is called the modulus of continuity of f on $[a,b]$.

FUZZY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

We consider the VIDE of the form,

$$y'(t) = f\left(t, y(t), \lambda\int_a^t K(t,s)y(s)ds\right), \quad a\le t\le b,$$
$$y(t_0) = y_0, \tag{3.1}$$

where $\lambda > 0$, $K(t,s)$ is an arbitrary kernel function over $S = \{(t,s): a\le s\le t\le b\}$ and $f\left(t, y(t), \lambda\int_a^t K(t,s)y(s)ds\right)$ is a continuous fuzzy function on the interval $[a,b]$. We assume that $\lambda > 0$. In order to design a numerical scheme for solving Eqn (3.1), we write the parametric form of the given Eqn (3.1) as follows:

$$\underline{y}'(t;\alpha) = \underline{f}\left(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha), \lambda\int_a^t \underline{U}(s;\alpha)ds\right),$$
$$\overline{y}'(t;\alpha) = \overline{f}\left(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha), \lambda\int_a^t \overline{U}(s;\alpha)ds\right), \tag{3.2}$$

where

$$\underline{U}(s;\alpha) = \underline{K(t,s)y(s;\alpha)} = \begin{cases} K(t,s)\underline{y}(s;\alpha), & K(t,s)\ge 0,\\ K(t,s)\overline{y}(s;\alpha), & K(t,s)<0\end{cases}$$

and

$$\overline{U}(s;\alpha) = \overline{K(t,s)y(s;\alpha)} = \begin{cases} K(t,s)\overline{y}(s;\alpha), & K(t,s)\ge 0,\\ K(t,s)\underline{y}(s;\alpha), & K(t,s)<0.\end{cases}$$

We can see that (3.2) is a system of FVIDE for each $0\le\alpha\le 1$ and $a\le t\le b.$

RUNGE-KUTTA METHOD FOR FUZZY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

The exact and approximate solutions at $t_{n,}$ $0\le n\le N$ are denoted by $[F(t)]^\alpha = [\underline{F}(t;\alpha), \overline{F}(t;\alpha)]$ and $[y(t)]^\alpha = [\underline{y}(t;\alpha), \overline{y}(t;\alpha)]$ respectively. We replace the interval $[a,b]$. by a set of discrete equally spaced

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grid points $a = t_0 < t_1 \ldots < t_N = b$ at which the exact solution $[F(t)]^{\alpha} = [\underline{F}(t;\alpha), \overline{F}(t;\alpha)]$ is approximated by some $[y(t)]^{\alpha} = [\underline{y}(t;\alpha), \overline{y}(t;\alpha)]$ respectively. The grid points at which the solution is calculated are $t_N = t_0 + nh, \quad h = \dfrac{b-a}{N}; 1 \le n \le N.$ The Runge-Kutta method for FVIDE is given by the formula

$$\underline{y}(t_{n+1};\alpha) = \min\left\{ \begin{array}{l} \underline{y}(t_n;\alpha) + h\sum_{i=1}^{m} b_i \underline{f}(t_n + c_i h, u, v+w) \Big| u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)], \\ v \in [\underline{z}_n(t_n + c_i h;\alpha), \overline{z}_n(t_n + c_i h;\alpha)], w \in [\underline{Z}_i(t_n;\alpha), \overline{Z}_i(t_n;\alpha)] \end{array} \right\},$$

$$\overline{y}(t_{n+1};\alpha) = \max\left\{ \begin{array}{l} \overline{y}(t_n;\alpha) + h\sum_{i=1}^{m} b_i \overline{f}(t_n + c_i h, u, v+w) \Big| u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)], \\ v \in [\underline{z}_n(t_n + c_i h;\alpha), \overline{z}_n(t_n + c_i h;\alpha)], w \in [\underline{Z}_i(t_n;\alpha), \overline{Z}_i(t_n;\alpha)] \end{array} \right\}, \tag{4.1}$$

where

$$\underline{Y}_i(t_n;\alpha) = \min\left\{ \begin{array}{l} \underline{y}(t_n;\alpha) + h\sum_{j=1}^{m} a_{ij} \underline{f}(t_n + c_j h, u, v+w) \Big| u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)], \\ v \in [\underline{z}_n(t_n + c_i h;\alpha), \overline{z}_n(t_n + c_i h;\alpha)], w \in [\underline{Z}_i(t_n;\alpha), \overline{Z}_i(t_n;\alpha)] \end{array} \right\},$$

$$\overline{Y}_i(t_n;\alpha) = \max\left\{ \begin{array}{l} \overline{y}(t_n;\alpha) + h\sum_{j=1}^{m} a_{ij} \overline{f}(t_n + c_j h, u, v+w) \Big| u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)], \\ v \in [\underline{z}_n(t_n + c_i h;\alpha), \overline{z}_n(t_n + c_i h;\alpha)], w \in [\underline{Z}_i(t_n;\alpha), \overline{Z}_i(t_n;\alpha)] \end{array} \right\},$$

$$\underline{Z}_i(t_n;\alpha) = \min\left\{ h\sum_{j=1}^{m} \overline{a}_{ij} K(t_n + d_{ij}h, s_n + c_j h)u \mid u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)] \right\},$$

$$\overline{Z}_i(t_n;\alpha) = \max\left\{ h\sum_{j=1}^{m} \overline{a}_{ij} K(t_n + d_{ij}h, s_n + c_j h)u \mid u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)] \right\}, \qquad (i = 1,\ldots,m).$$

We now discuss Runge-Kutta methods for FVIDE where the lag term in Eqn (4.1) is discretized by

$$\underline{z}_i(t_n + c_i h;\alpha) = \min\left\{ h\sum_{i=1}^{m}\sum_{j=1}^{n-1} b_i K(t_n + c_i h, s_j + c_i h)u \mid u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)] \right\},$$

$$\overline{z}_i(t_n + c_i h;\alpha) = \max\left\{ h\sum_{i=1}^{m}\sum_{j=1}^{n-1} b_i K(t_n + c_i h, s_j + c_i h)u \mid u \in [\underline{Y}_i(t_n;\alpha), \overline{Y}_i(t_n;\alpha)] \right\}, \qquad (i = 1,\ldots,m).$$

we assume $c_i = \sum_{j=1}^{m} a_{ij} (i, j = 1,\ldots,m)$ and require that $d_{ij} \ge c_j$ whenever $b_i \ne 0$ to ensure that the arguments of S in Eqn (4.1) are in the domain S.

The non-zero constant $c_i, b_i a_{ij}, \overline{a}_{ij}$ in the explicit Runge-Kutta(RK) method $(m = 2)$ for FVIDE are

$$c_1 = 0, c_2 = 1, \quad b_1 = b_2 = \frac{1}{2}, \quad a_{11} = a_{22} = 0, a_{21} = 1, \quad \overline{a}_{11} = \frac{1}{3}, \overline{a}_{21} = \frac{2}{3}, \overline{a}_{22} = 0$$

and we have

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$$\underline{Y}_1(t_n;\alpha)=\min\{u \mid u\in[\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha)]\},$$
$$\overline{Y}_1(t_n;\alpha)=\max\{u \mid u\in[\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha)],\},$$

$$\underline{Z}_1(t_n;\alpha)=\min\left\{\frac{h}{3}K(t_n+h,s_n)u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$
$$\overline{Z}_1(t_n;\alpha)=\max\left\{\frac{h}{3}K(t_n+h,s_n)u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$
$$\underline{Y}_2(t_n;\alpha)=\min\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+h\underline{f}(t_n,u,v+w)\mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\\ v\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],w\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)]\end{array}\right\},$$
$$\overline{Y}_2(t_n;\alpha)=\max\left\{\begin{array}{l}\overline{y}(t_n;\alpha)+h\overline{f}(t_n,u,v+w)\mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\\ v\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],w\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)]\end{array}\right\},$$
$$\underline{Z}_2(t_n;\alpha)=\min\left\{\frac{2h}{3}K(t_n+h,s_n)u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$
$$\overline{Z}_2(t_n;\alpha)=\max\left\{\frac{2h}{3}K(t_n+h,s_n)u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$

Now we define

$$\underline{y}(t_{n+1};\alpha)=\min\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+\frac{h}{2}\underline{f}(t_n,u_1,v_1)+\frac{h}{2}\underline{f}(t_n+h,u_2,v_2+w)\mid\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\ u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ v_1\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],v_2\in[\underline{z}_n(t_n+h;\alpha),\overline{z}_n(t_n+h;\alpha)]\\ w\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)]\end{array}\right\},$$
$$\overline{y}(t_{n+1};\alpha)=\max\left\{\begin{array}{l}\overline{y}(t_n;\alpha)++\frac{h}{2}\overline{f}(t_n,u_1,v_1)+\frac{h}{2}\overline{f}(t_n+h,u_2,v_2+w)\mid\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\ u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ v_1\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],v_2\in[\underline{z}_n(t_n+h;\alpha),\overline{z}_n(t_n+h;\alpha)]\\ w\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)\end{array}\right\},\quad(4.2)$$

The non-zero constant $c_i, b_i a_{ij}, \overline{a}_{ij}$ in the explicit 4-stage Runge-Kutta(RK) method $(m=4)$ for FVIDE which are satisfy $a_{ij}=\overline{a}_{ij}$.

$$c_1=0, c_2=c_3=\frac{1}{2}, c_4=1,\quad a_{11}=a_{22}=a_{33}=a_{41}=a_{42}=a_{44}=0, a_{21}=\frac{1}{2}, a_{31}=\frac{4}{9}, a_{32}=\frac{1}{18}, a_{43}=1,$$
$$b_1=\frac{1}{6}, b_2=b_3=\frac{1}{3}, b_4=\frac{1}{6}$$

and we have

$$\underline{Y}_1(t_n;\alpha)=\min\{u \mid u\in[\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha)]\},$$
$$\overline{Y}_1(t_n;\alpha)=\max\{u \mid u\in[\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha)]\},$$
$$\underline{Z}_1(t_n;\alpha)=0,\quad \overline{Z}_1(t_n;\alpha)=0,$$

$$\underline{Y}_2(t_n;\alpha)=\min\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+\frac{h}{2}\underline{f}(t_n,u,v_1+w)\Big| u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\\ w\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)]\end{array}\right\},$$
$$\overline{Y}_2(t_n;\alpha)=\max\left\{\begin{array}{l}\overline{y}(t_n;\alpha)+\frac{h}{2}\overline{f}(t_n,u,v_1+w)\Big| u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],\\ w\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)]\end{array}\right\},$$
$$\underline{Z}_2(t_n;\alpha)=\min\left\{\frac{h}{2}K(t_n+\frac{h}{2},s_n+\frac{h}{2})u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$
$$\overline{Z}_2(t_n;\alpha)=\max\left\{\frac{h}{2}K(t_n+\frac{h}{2},s_n+\frac{h}{2})u \mid u\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)]\right\},$$
$$\underline{Y}_3(t_n;\alpha)=\min\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+\frac{4h}{9}\underline{f}(t_n,u_1,v_1+w_1)+\frac{h}{18}\underline{f}(t_n+\frac{h}{2},u_2,v_2+w_2)\Big|\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ w_1\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)],w_2\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)]\end{array}\right\},$$
$$\overline{Y}_3(t_n;\alpha)=\max\left\{\begin{array}{l}\overline{y}(t_n;\alpha)+\frac{4h}{9}\overline{f}(t_n,u_1,v_1+w_1)+\frac{h}{18}\overline{f}(t_n+\frac{h}{2},u_2,v_2+w_2)\Big|\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ w_1\in[\underline{Z}_1(t_n;\alpha),\overline{Z}_1(t_n;\alpha)],w_2\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)]\end{array}\right\},$$
$$\underline{Z}_3(t_n;\alpha)=\min\left\{\begin{array}{l}\frac{4h}{9}K(t_n,s_n)u_1+\frac{1}{18}K(t_n+\frac{h}{2},s_n+\frac{h}{2})u_2\Big|\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)]\end{array}\right\},$$
$$\overline{Z}_3(t_n;\alpha)=\max\left\{\begin{array}{l}\frac{4h}{9}K(t_n,s_n)u_1+\frac{1}{18}K(t_n+\frac{h}{2},s_n+\frac{h}{2})u_2\Big|\\ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)]\end{array}\right\},$$
$$\underline{Y}_4(t_n;\alpha)=\min\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+h\underline{f}(t_n+\frac{h}{2},u,v_3+w)\Big|\\ u\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)],w\in[\underline{Z}_3(t_n;\alpha),\overline{Z}_3(t_n;\alpha)]\end{array}\right\},$$
$$\overline{Y}_4(t_n;\alpha)=\min\left\{\begin{array}{l}\overline{y}(t_n;\alpha)+h\overline{f}(t_n+\frac{h}{2},u,v_3+w)\Big|\\ u\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)],w\in[\underline{Z}_3(t_n;\alpha),\overline{Z}_3(t_n;\alpha)]\end{array}\right\},$$
$$\underline{Z}_4(t_n;\alpha)=\min\left\{hK(t_n+\frac{h}{2},s_n+\frac{h}{2})u\Big| u\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)]\right\},$$

$$\overline{Z}_4(t_n;\alpha)=\max\left\{hK(t_n+\frac{h}{2},s_n+\frac{h}{2})u \middle| u\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)]\right\}$$

where

$$v_1=[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)]=\left[\frac{h}{6}K(t_n,s_n)\underline{Y}_1(t_n;\alpha),\frac{h}{6}K(t_n,s_n)\overline{Y}_1(t_n;\alpha)\right],$$

$$v_2=[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)]=\left[\frac{h}{3}K(t_n+\frac{h}{2},s_n+\frac{h}{2})\underline{Y}_2(t_n;\alpha),\frac{h}{3}K(t_n+\frac{h}{2},s_n+\frac{h}{2})\overline{Y}_2(t_n;\alpha)\right],$$

$$v_3=[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)]=\left[\frac{h}{3}K(t_n+\frac{h}{2},s_n+\frac{h}{2})\underline{Y}_3(t_n;\alpha),\frac{h}{3}K(t_n+\frac{h}{2},s_n+\frac{h}{2})\overline{Y}_3(t_n;\alpha)\right],$$

$$v_4=[\underline{z}_n(t_n+h;\alpha),\overline{z}_n(t_n+h;\alpha)]=\left[\frac{h}{6}K(t_n+h,s_n+h)\underline{Y}_4(t_n;\alpha),\frac{h}{6}K(t_n+h,s_n+h)\overline{Y}_4(t_n;\alpha)\right].$$

Now we define

$$\underline{y}(t_{n+1};\alpha)$$
$$=\left\{\begin{array}{l}\underline{y}(t_n;\alpha)+\frac{h}{6}\underline{f}(t_n,u_1,v_1)+\frac{h}{3}\underline{f}(t_n+\frac{h}{2},u_2,v_2+w_1)+\frac{h}{3}\underline{f}(t_n+\frac{h}{2},u_3,v_3+w_2)\\ +\frac{h}{6}\underline{f}(t_n+h,u_4,v_4+w_3) \big| \ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ u_3\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)],u_4\in[\underline{Y}_4(t_n;\alpha),\overline{Y}_4(t_n;\alpha)],v_1\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],\\ v_2\in[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)],v_3\in[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)],\\ v_4\in[\underline{z}_n(t_n+h;\alpha),\overline{z}_n(t_n+h;\alpha)],w_1\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)],\\ w_2\in[\underline{Z}_3(t_n;\alpha),\overline{Z}_3(t_n;\alpha)],w_3\in[\underline{Z}_4(t_n;\alpha),\overline{Z}_4(t_n;\alpha)],\end{array}\right\}$$

$$\overline{y}(t_{n+1};\alpha)$$
$$=\left\{\begin{array}{l}\overline{y}(t_n;\alpha)+\frac{h}{6}\overline{f}(t_n,u_1,v_1)+\frac{h}{3}\overline{f}(t_n+\frac{h}{2},u_2,v_2+w_1)+\frac{h}{3}\overline{f}(t_n+\frac{h}{2},u_3,v_3+w_2)\\ +\frac{h}{6}\overline{f}(t_n+h,u_4,v_4+w_3) \big| \ u_1\in[\underline{Y}_1(t_n;\alpha),\overline{Y}_1(t_n;\alpha)],u_2\in[\underline{Y}_2(t_n;\alpha),\overline{Y}_2(t_n;\alpha)],\\ u_3\in[\underline{Y}_3(t_n;\alpha),\overline{Y}_3(t_n;\alpha)],u_4\in[\underline{Y}_4(t_n;\alpha),\overline{Y}_4(t_n;\alpha)],v_1\in[\underline{z}_n(t_n;\alpha),\overline{z}_n(t_n;\alpha)],\\ v_2\in[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)],v_3\in[\underline{z}_n(t_n+\frac{h}{2};\alpha),\overline{z}_n(t_n+\frac{h}{2};\alpha)],\\ v_4\in[\underline{z}_n(t_n+h;\alpha),\overline{z}_n(t_n+h;\alpha)],w_1\in[\underline{Z}_2(t_n;\alpha),\overline{Z}_2(t_n;\alpha)],\\ w_2\in[\underline{Z}_3(t_n;\alpha),\overline{Z}_3(t_n;\alpha)],w_3\in[\underline{Z}_4(t_n;\alpha),\overline{Z}_4(t_n;\alpha)],\end{array}\right\} \quad (4.3)$$

Lemma 4.1.

Let a sequence of numbers $\{\xi_i\}_{n=0}^{N}$ satisfy

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$$\xi_n \le A|\xi_{n-1}| + \sum_{i=0}^{n-1}|\xi_i| + C,$$

with $A > 1$, positive B, C and $\xi_0 = 0$. Then

$$\xi_n \le \frac{(A+nB)^n - 1}{A-1}C. \tag{4.4}$$

Now, the first order approximation of $F'(t)$ is given by [14]

$$F'(t;\alpha) = \frac{F(t+h;\alpha) - F(t;\alpha)}{h}. \tag{4.5}$$

By virtue of equation (4.5) we obtain

$$y(t_{n+1}) = y(t_n) + hf\left(t_n, \sum_{j=0}^{n-1}\frac{b-a}{2n}\left[K(t_n,t_j)y(t_j) + K(t_n,t_{j+1})y(t_{j+1})\right]\right); \tag{4.6}$$

$$y(t_n) = y(t_0) = F_0; \quad n = 0,1,\ldots, \tag{4.7}$$

By Theorem 2.7 in [21] we may replace (4.5) by the equivalent system

$$\underline{y}(t_{n+1};\alpha) = \underline{y}(t_n;\alpha) + h\left[\underline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix}K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\right];$$

$$\overline{y}(t_{n+1};\alpha) = \overline{y}(t_n;\alpha) + h\left[\overline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix}K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\right]; \tag{4.8}$$

$$\underline{y}(t_n;\alpha) = \underline{y}(t_0;\alpha) = \underline{F}_0; \quad \overline{y}(t_n;\alpha) = \overline{y}(t_0;\alpha) = \overline{F}_0; \quad n = 0,1,\ldots,$$

Let $\underline{f}(t,s,u,v)$ and $\overline{f}(t,s,u,v)$ be functions $\underline{f}$ and $\overline{f}$ of Eqn (4.7), where u and v are constants and $u \le v$. In other words, $\underline{f}(t,s,u,v)$ and $\overline{f}(t,s,u,v)$ are obtained by substituting $y = (u,v)$ in Eqn (4.7). The domain where $\underline{f}$ and $\overline{f}$ are defined

$$B = \{(t,s,u,v) \mid a \le s, \quad t \le b, \quad -\infty < v < \infty, \quad -\infty < u \le v\}. \tag{4.9}$$

Theorem 4.1.
Let $\underline{f}(t,s,u,v)$ and $\overline{f}(t,s,u,v)$ belong to $C^4(B)$. Let the partial derivatives of $\underline{f}, \overline{f}$ be bounded over B and also $D(F_n, y_n) = \max_{0\le i\le N}\{D(F_i, y_i)\}$. Then, for arbitrary fixed α, $0 \le \alpha \le 1$,

$$\lim_{h\to 0}\underline{y}(t_n;\alpha) = \underline{F}(t_n;\alpha); \quad \lim_{h\to 0}\overline{y}(t_n;\alpha) = \overline{F}(t_n;\alpha).$$

Proof: Let

$$\underline{F}(t_{n+1};\alpha) = \underline{F}(t_n;\alpha) + h\left[\underline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{bmatrix}\right)\right] + O(h^5),$$

$$\overline{F}(t_{n+1};\alpha) = \overline{F}(t_n;\alpha) + h\left[\overline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{bmatrix}\right)\right] + O(h^5)$$

and

$$\underline{y}(t_{n+1};\alpha) = \underline{y}(t_n;\alpha) + h\left[\underline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\right] + O(h^5),$$

$$\overline{y}(t_{n+1};\alpha) = \overline{y}(t_n;\alpha) + h\left[\overline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\right] + O(h^5).$$

Consequently,

$$\underline{F}(t_{n+1};\alpha) - \underline{y}(t_{n+1};\alpha) = \underline{F}(t_n;\alpha) - \underline{y}(t_n;\alpha) + h\left[\begin{array}{l} \underline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{bmatrix}\right) \\ -\underline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\end{array}\right] + O(h^5),$$

$$\overline{F}(t_{n+1};\alpha) - \overline{y}(t_{n+1};\alpha) = \overline{F}(t_n;\alpha) - \overline{y}(t_n;\alpha) + h\left[\begin{array}{l} \overline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{bmatrix}\right) \\ -\overline{f}\left(t_n, \underline{y}(t_n;\alpha), \overline{y}(t_n;\alpha), \sum_{j=0}^{n-1}\frac{b-a}{2n}\begin{bmatrix} K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha)) \\ +K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{bmatrix}\right)\end{array}\right] + O(h^5),$$

now rearrange the above equation, we get

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$$\underline{F}(t_{n+1};\alpha)-\underline{y}(t_{n+1};\alpha)$$

$$=\underline{F}(t_n;\alpha)-\underline{y}(t_n;\alpha)+h\left[\begin{array}{l}\underline{f}(t_n,\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha))+\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[\begin{array}{l}K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha))\\+K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{array}\right]\\-\underline{f}(t_n,\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha))-\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[\begin{array}{l}K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha))\\+K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{array}\right]\end{array}\right]+O(h^5),$$

$$\overline{F}(t_{n+1};\alpha)-\overline{y}(t_{n+1};\alpha)$$

$$=\overline{F}(t_n;\alpha)-\overline{y}(t_n;\alpha)+h\left[\begin{array}{l}\overline{f}(t_n,\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha))+\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[\begin{array}{l}K(t_n,t_j)(\underline{F}(t_j;\alpha),\overline{F}(t_j;\alpha))\\+K(t_n,t_{j+1})(\underline{F}(t_{j+1};\alpha),\overline{F}(t_{j+1};\alpha))\end{array}\right]\\-\overline{f}(t_n,\underline{y}(t_n;\alpha),\overline{y}(t_n;\alpha))-\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[\begin{array}{l}K(t_n,t_j)(\underline{y}(t_j;\alpha),\overline{y}(t_j;\alpha))\\+K(t_n,t_{j+1})(\underline{y}(t_{j+1};\alpha),\overline{y}(t_{j+1};\alpha))\end{array}\right]\end{array}\right]+O(h^5),$$

Denote $W(t_{n+1})=\underline{F}(t_{n+1};\alpha)-\underline{y}(t_{n+1};\alpha)$ and $V(t_{n+1})=\overline{F}(t_{n+1};\alpha)-\overline{y}(t_{n+1};\alpha)$. Then

$$|W(t_{n+1})|\le|W(t_n)|+h\left[\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[2L\max\{|W(t_j),V(t_j)|\}2L\max\{|W(t_{j+1}),V(t_{j+1})|\}\right]\right]+O(h^5),$$

$$|V(t_{n+1})|\le|V(t_n)|+h\left[\sum_{j=0}^{n-1}\frac{b-a}{2n}\left[2L\max\{|W(t_j),V(t_j)|\}2L\max\{|W(t_{j+1}),V(t_{j+1})|\}\right]\right]+O(h^5),$$

where $L>0$ is a bound for the partial derivatives of $\underline{f}$, $\overline{f}$. Thus, we have

$$|W(t_{n+1})|\le|W(t_n)|+2nLh\frac{b-a}{2n}D(F(t_n),y(t_n))+O(h^5),$$

$$|V(t_{n+1})|\le|V(t_n)|+2nLh\frac{b-a}{2n}D(F(t_n),y(t_n))+O(h^5),$$

Since $n=0, W(t_n)=V(t_n)=W(t_0)=V(t_0)=0,$ we obtain

$$|W(t_{n+1})|\le 2(n+1)Lh\frac{b-a}{2n}D(F(t_n),y(t_n))+O(h^5),$$

$$|V(t_{n+1})|\le 2(n+1)Lh\frac{b-a}{2n}D(F(t_n),y(t_n))+O(h^5),$$

and if $h\to 0,$ we get $D(F(t_n),y(t_n))\to 0.$

NUMERICAL EXAMPLES

In this section, we give the approximate solution of linear and nonlinear FVIDE by using Runge-Kutta method of order four.

Example 5.1.

Consider the following linear FVIDE

$$\underline{y}'(t;\alpha)=(0.5+0.5\alpha)(e^t-t)+\int_0^t st\,\underline{y}(s;\alpha)ds,$$

$$\overline{y}'(t;\alpha)=(2-\alpha)(e^t-t)+\int_0^t st\,\overline{y}(s;\alpha)ds,$$

$$\underline{y}(0)=0.5+0.5\alpha;\ \overline{y}(0)=2-\alpha;\quad 0\le\alpha\le 1,\quad 0\le s\le t,\quad t\in[0,1].$$

The exact solution is given by $\underline{Y}(t;\alpha)=(0.5+0.5\alpha)e^t$; $\overline{Y}(t;\alpha)=(2-\alpha)e^t$. The exact and obtained approximate solutions of FVIDE are compared in Figure 1.

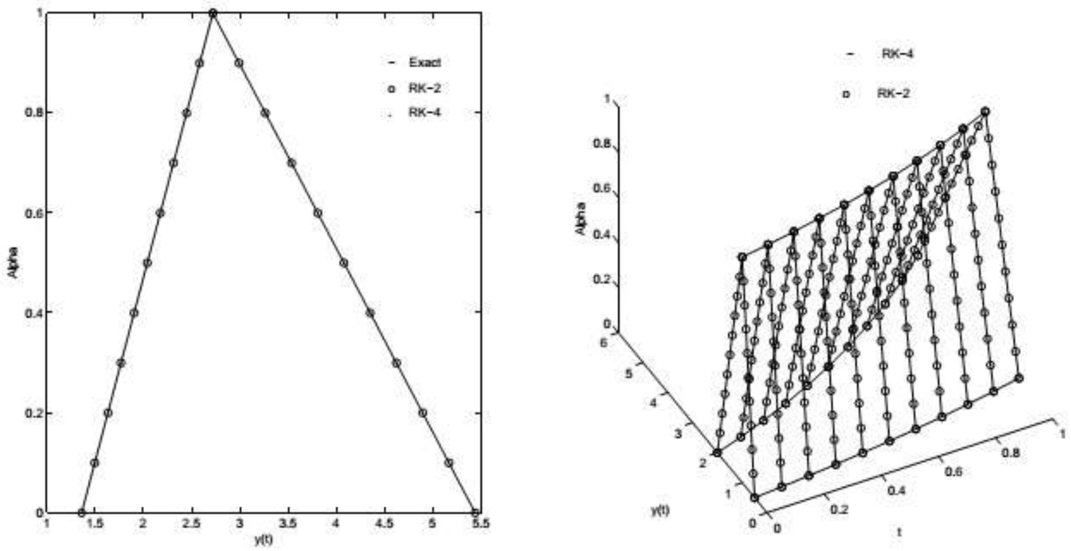

Fig 1 : The exact and approximate solutions for FVIDE- Runge-Kutta method of order 4

Example 5.2.

We consider the following linear FVIDE

$$\underline{y}'(t;\alpha)=(\alpha-1)+\int_0^t \underline{y}(s;\alpha)ds,$$

$$\overline{y}'(t;\alpha)=(1-\alpha)+\int_0^t \overline{y}(s;\alpha)ds,$$

$$\underline{y}(0)=0;\ \overline{y}(0)=0;\quad 0\le\alpha\le1,\quad 0\le s\le t,\quad t\in[0,1].$$

The exact solution is given by $\underline{Y}(t;\alpha)=(\alpha-1)\sinh(t)$; $\overline{Y}(t;\alpha)=(1-\alpha)\sinh(t)$.

The exact and obtained approximate solutions of FVIDE are compared in Figure 2.

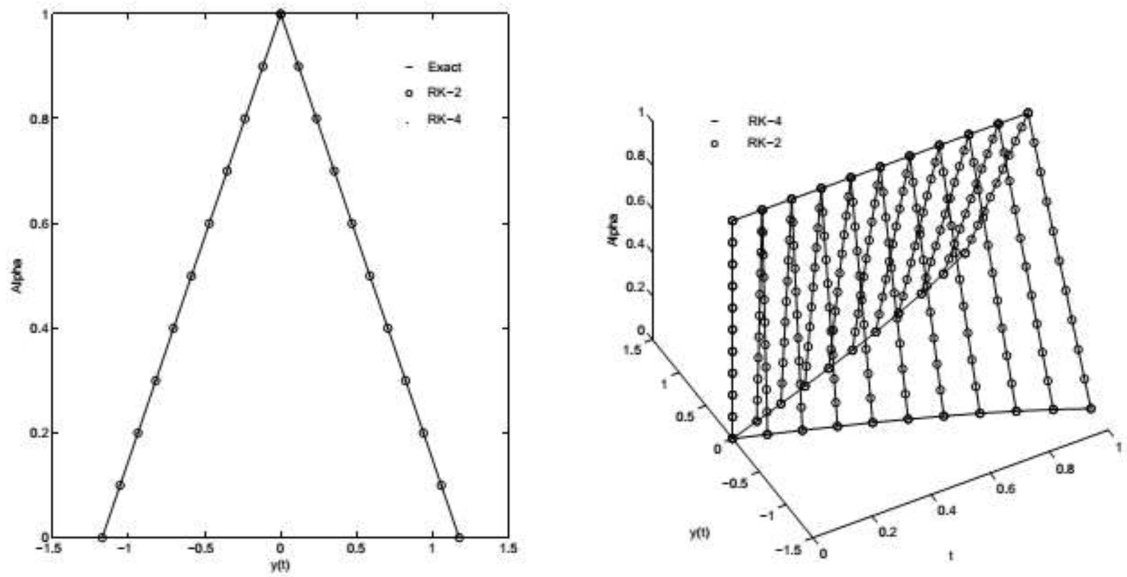

Fig 2 : The exact and approximate solutions for FVIDE- Runge-Kutta method of order 4

CONCLUSION

In this study, we used Runge-Kutta method of order four for finding the approximate solutions of FVIDE. The efficiency of this method is illustrated by solving some linear and nonlinear FVIDE. We feel that this work may help to narrow the existing gap between the theoretical research on FVIDE and practical applications used in desiring various fuzzy dynamical systems.

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REFERENCES

[1] S. Abbasbandy, T. Allahviranloo, “Numerical solutions of fuzzy differential equations by Taylor method” in Computational Methods in Applied Mathematics, Vol.2, pp. 113-124, 2002.
[2] S. Abbasbandy, T. Allahviranloo, “Numerical solution of fuzzy differential equation by Runge-Kutta method”, Nonlinear Studies, Vol.11, pp. 117-129, 2004.
[3] H. Brunner, E. Hairer and P. Norsett, “Runge-Kutta theory for Volterra integral equations of the second kind”, Mathamatics of Computation, Vol.39, pp. 147-163, 1982.
[4] J. J. Buckley, T. Feuring, Y. Hayashi, “Fuzzy integral equations”, IEEE, pp. 319-323.
[5] W. Congxin, M. Ming, “On the integrals, series and integral equations of fuzzy set-valued functions”, Journal of Harbin Instition Technology, Vol.21, pp. 9-11, 1990.
[6] D. Dubois, H. Prade, “Towards fuzzy differential calculus, Fuzzy Sets and Systems, Vol.8, pp. 1-7, 105-116, 225-233, 1982.
[7] M. Friedman, M. Ma, A. Kandel, “On fuzzy integral equations”, Fundamenta Informaticae, 37, pp. 89-99, 1999.
[8] M. Friedman, M. Ma, A. Kandel, “Numerical solutions of fuzzy differential and integral equations”, Fuzzy Sets and Systems, Vol.106, pp. 35-48, 1999.
[9] R. Goetschel, W. Voxman, “Elementary calculus”, Fuzzy Sets and Systems, Vol.18, pp. 31-43, 1986.
[10] H. Hochstadt, “Integral Equations”, Wiley, New York, 1973.
[11] O. Kaleva, “Fuzzy differential equations”, Fuzzy Sets and Systems, Vol.24, pp. 301-317, 1987.
[12] V. Lakshmikantham, R. N. Mohapatra, “Theory of fuzzy differential equations and inclusions”, Taylor and Francis, London, 2003.
[13] Ch. Lubich, “Runge-Kutta theory for Volterra integrodifferential equations”, Numerische Mathematik, Vol.40, pp. 119-135, 1982.

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CODEN: IJESS7

[14] M. Ma, M. Friedman, A. Kandel, "Numerical solutions of fuzzy differential equations", Fuzzy Sets and Systems, Vol.105, pp. 133-138, 1999.

[15] M. Ma, M. Friedman, A. Kandel, "Numerical solutions of fuzzy differential and integral equations", Fuzzy Sets and Systems, Vol.106, pp. 35-48, 1999.

[16] Mashaallah Matinfar, Mojtaba Ghanbari, Rahele Nuraei, "Numerical solution of linear fuzzy Volterra integro-differential equations by variational iteration method", Journal of Intelligent and Fuzzy systems, 24, pp. 575-586, 2013.

[17] S. Hajighasemi, T. Allahviranloo, M. Khezerloo, M. Khorasany, S. Salahshour, "Existence and Uniqueness of solutions of fuzzy Volterra integro-differential equations", IPMU Part II, CCIS 81, Springer-Verlag, pp. 491-500, 2010.

[18] G. J. Kril, U. S. Clair and B. Yuan, "Fuzzy set theory: foundations and applications", Prentice Hall Inc, 1997.

[19] K. Maleknejad, M. Shahrezaee, "Using Runge-Kutta method for numerical solution of the system of Volterra integral equation", Applied Mathematics and Computation, Vol.149, pp. 399-410, 2004.

[20] J. Mordeson, W. Newman, "Fuzzy integral equations", Information Sciences, Vol.87, pp. 215-229, 1995.

[21] O. Kaleva, "Fuzzy differential equations", Fuzzy Sets and Systems, Vol.24(3), pp. 301-317, 1987.

[22] J. Y. Park, J.U.Jeong, "A note on fuzzy integral equations", Fuzzy Sets and Systems, Vol.108, pp. 193-200, 1999.

[23] Peter Linz, "Analytical and numerical methods for Volterra Integral equations", Davis, California, 1985.

[24] Phil Diamond, "Theory and applications of fuzzy Volterra integral equations", IEEE Transactions on Fuzzy Systems, Vol.10, pp. 97-102, 2002.

[25] M. L. Puri, D. A. Ralescu, "Fuzzy random variables", Journal of Mathematical Analysis and Application, Vol.114, pp. 409-422, 1986.

[26] D. Ralescu, G. Adams, "The fuzzy integral", Journal of Mathematical Analysis and Applications Vol.75, pp. 562-570, 1980.

[27] S. Seikkala, "On the fuzzy initial value problem", Fuzzy Sets and Systems, Vol.24(3) pp. 319-330, 1987.

[28] P. V. Subrahmanyam, S. K. Sudarsanam, "A note on fuzzy Volterra integral equations", Fuzzy Sets and Systems, Vol.81, pp. 237-240, 1996.

[29] W. Congxin and M. Ming, "Embedding problem of fuzzy number space", Fuzzy Sets and Systems, Vol.45(2), pp. 189-202, 1992.

[30] L. A. Zadeh, "Fuzzy sets", Information and Control, Vol.8, pp. 338-353, 1965.